WHICH MULTIPLIER ALGEBRAS ARE W^*-ALGEBRAS?

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Abstract. We consider the question of when the multiplier algebra $M(\mathcal{A})$ of a C^*-algebra $\mathcal{A}$ is a W^*-algebra, and show that it holds for a stable C^*-algebra exactly when it is a C^*-algebra of compact operators. This implies that if for every Hilbert C^*-module E over a C^*-algebra $\mathcal{A}$, the algebra $B(E)$ of adjointable operators on E is a W^*-algebra, then $\mathcal{A}$ is a C^*-algebra of compact operators.

Also we show that a unital C^*-algebra $\mathcal{A}$ which is Morita equivalent to a W^*-algebra must be a W^*-algebra.

1. Introduction

The main theme of this paper is around the question of when the multiplier algebra $M(\mathcal{A})$ of a C^*-algebra $\mathcal{A}$ is a W^*-algebra? For separable C^*-algebras, it holds exactly when $\mathcal{A}$ is a C^*-algebra of compact operators [2, Theorem 2.8]. For general C^*-algebras, we get two partial results in this direction. First we give an affirmative answer for stable C^*-algebras and deduce that if for every Hilbert C^*-module E over $\mathcal{A}$, the algebra $B(E)$ of adjointable operators on E is a W^*-algebra, then $\mathcal{A}$ is a C^*-algebra of compact operators. This is related to our question (with a much stronger assumption) as for $E = A$ with its canonical Hilbert $\mathcal{A}$-module structure, $B(E) = M(\mathcal{A})$. Second we show that if $M(\mathcal{A})$ is Morita equivalent to a W^*-algebra, then it is a W^*-algebra. This is also related to our question, as if $\mathcal{A}$ is a C^*-algebra of compact operators, then $M(\mathcal{A})$ is a W^*-algebra.

The two partial answers take into account the notions of Hilbert C^*-algebras and Morita equivalence which are somewhat historically related. In 1953, Kaplansky introduces Hilbert C^*-modules to prove that derivations of type I AW^*-algebras are inner. Twenty years later, Hilbert C^*-modules appeared in the pioneering work of Rieffel [19],

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where he employed them to study (strong) Morita equivalence of C^*-algebras. Paschke studied Hilbert C^*-modules as a generalization of Hilbert spaces [16].

Hilbert C^*-modules and Hilbert spaces differ in many aspects, such as existence of orthogonal complements for submodules (subspaces), self duality, existence of orthogonal basis, adjointability of bounded operators, etc. However, when $\mathcal{A}$ is a C^*-algebra of compact operators, then Hilbert $\mathcal{A}$-modules behave like Hilbert spaces in having the above properties. Indeed these properties characterize C^*-algebras of compact operators [5, 10, 14, 20].

2. C^*-algebras of compact operators

In this section we give some characterizations of C^*-algebras of compact operators using properties of multiplier algebras. We also show that these are characterized as C^*-algebras $\mathcal{A}$ for which the algebra $B(E)$ of all adjointable operators is a W^*-algebra, for any Hilbert $\mathcal{A}$-module E.

Definition 2.1. A C^*-algebra $\mathcal{A}$ is called a C^*-algebra of compact operators if there exists a Hilbert space H and a (not necessarily surjective) $*$-isomorphism from $\mathcal{A}$ to $K(H)$, where $K(H)$ denotes the space of compact operators on H.

This is exactly how Kaplansky characterized C*-algebras that were dual rings [11, Theorem 2.1, p. 222] (see also [1]).

Theorem 2.2. *For a C^*-algebra $\mathcal{A}$, the following are equivalent:*

(i) $\mathcal{A}$ is a C^*-algebra of compact operators.

(ii) The strict topology on the unit ball of $M(\mathcal{A})$ is the same as the strong*-topology (viewing $M(\mathcal{A}) \subseteq \mathcal{A}^{**}$, the second dual of $\mathcal{A}$).

Proof. Assume that (i) holds. Then $\mathcal{A} \cong c_0$-$\sum \bigoplus_\alpha K(H_\alpha)$. Let $a_\beta \to 0$ in the strict topology of the unit ball of $M(\mathcal{A}) \cong \ell^\infty$-$\sum \bigoplus_\alpha B(H_\alpha)$. Without loss of generality, we may assume that $a_\beta \geq 0$, for all β. Let $\eta \in \bigoplus_\alpha H_\alpha$ be a unit vector with $\eta_\alpha = 0$ except for finitely many α. Let p_α be the rank one projection onto the non-zero η_α and $p_\alpha = 0$, otherwise. Then $p = \sum p_\alpha \in \mathcal{A}$, thus $\|a_\beta p\| \to 0$. Therefore $\|a_\beta \eta\| \to 0$, and the same holds for any η in the unit ball of $\bigoplus_\alpha H_\alpha$, as $\{a_\beta\}$ is norm bounded. Hence $a_\beta \to 0$ in the strong* topology.

Conversely if $a_\beta \geq 0$ and $a_\beta \to 0$ in the strong* topology. As above, for any rank one projection $p \in \mathcal{A}$, $\|a_\beta p\| = \|pa_\beta\| \to 0$. Thus p can be replaced by any finite linear combination of such minimal projections, and this set is dense in $\mathcal{A}$. Since $\{a_\beta\}$ is norm bounded, $a_\beta \to 0$ in the strict topology. This shows that (i) implies (ii).

Now assume that (ii) holds. By [2, Theorem 2.8], we need only to prove that $M(\mathcal{A}) = \mathcal{A}^{**}$. For any positive element b in the unit ball of $\mathcal{A}^{**}$, there is a net $\{a_\beta\}$ in the unit ball of $\mathcal{A}$ that converges to b in strong* topology. Thus the net is strong* Cauchy, and hence convergent in the strict topology to an element of $M(\mathcal{A})$, as $M(\mathcal{A})$ is the completion of $\mathcal{A}$ in the strict topology [9, Theorem 3.6]. Therefore $b \in M(\mathcal{A})$, and we are done. □

Another characterization of C^*-algebras of compact operators could be obtained as a non unital version of the following result of J.A. Mingo in [15], where he investigates the multipliers of stable C^*-algebras.

Lemma 2.3. *Suppose that H is a separable infinite dimensional Hilbert space and $\mathcal{A}$ is a unital C^*-algebra such that the multiplier algebra $M(\mathcal{A} \otimes K(H))$ is a W^*-algebra. Then $\mathcal{A}$ is a finite dimensional C^*-algebra.*

We recall that a projection p in a C^*-algebra $\mathcal{A}$ is called finite dimensional if $p\mathcal{A}p$ is a finite dimensional C^*-algebra. To prove a non unital version of Mingo's result, we need some lemmas. The first lemma is well-known, see for instance [4, Corollary 1.2.37].

Lemma 2.4. *If $\mathcal{A}$ is a C^*-algebra and p is a projection in the multiplier algebra $M(\mathcal{A})$, then $M(p\mathcal{A}p) \cong pM(\mathcal{A})p$, as C^*-algebras.*

Lemma 2.5. *Let H be a Hilbert space and $\mathcal{A}$ be a C^*-algebra. If $\mathcal{A} \otimes K(H)$ is C^*-algebra of compact operators, then so is $\mathcal{A}$.*

Proof. Suppose not. Then there is an element $b \in \mathcal{A}^+$ such that the spectral projection $\xi_1(b)$ of b corresponding to $\{1\}$ is not finite dimensional in $\mathcal{A}$. Let q be a one-dimensional projection in $K(H)$. Then $(b \otimes q)^n$ is a decreasing sequence in the unit ball of the C*-algebra $\mathcal{A} \otimes K(H)$ of compact operators. By Theorem 2.2 it converges strictly, hence (because it is decreasing) in norm to $\xi_1(b) \otimes q \in \mathcal{A}$. Because $\mathcal{A} \otimes K(H)$ is a C^*-algebra of compact operators, the projection $\xi_1(b) \otimes q$ must be finite rank, but

$$(\xi_1(b) \otimes q)(\mathcal{A} \otimes K(H))(\xi_1(b) \otimes q) = \xi_1(b)\mathcal{A}\xi_1(b) \otimes qK(H))q,$$

and the dimension of $\xi_1(b)\mathcal{A}\xi_1(b)$ is not finite by our assumption about b. □

The next theorem is known for separable C^*-algebras [2], here we prove it with separability replaced by stability.

Theorem 2.6. *If $\mathcal{A}$ is a stable C^*-algebra such that the multiplier algebra $M(\mathcal{A})$ is a W^*-algebra, then $\mathcal{A}$ is a C^*-algebra of compact operators.*

Proof. In order for the C^*-algebra $\mathcal{A}$ to be a C^*-algebra of compact operators, it is necessary and sufficient that every positive element in $\mathcal{A}$ can be approximated by a finite linear combination of finite dimensional projections. Let a be a positive element in $\mathcal{A}$ and $0 \leq a \leq 1$. Since the multiplier algebra $M(\mathcal{A})$ is a W^*-algebra, we can define $p \in M(\mathcal{A})$ as the spectral projection of a, corresponding to an interval of the form $[s,t]$ where $0 < s < t$. It suffices to show that $p\mathcal{A}p$ is finite dimensional. Let $g : [0,1] \to [0,1]$ be a continuous function vanishing at 0, such that $g(r) = 1$ for all $r \in [s,t]$. Then $g(a) \in A$ and $g(a)p = p$. Hence $p \in A$.

Now let H be a separable infinite dimensional Hilbert space. Since $\mathcal{A}$ is a stable C^*-algebra, $M(\mathcal{A}) = M(\mathcal{A} \otimes K(H))$ is a W^*-algebra and by Lemma 2.4,

$$\begin{aligned} M(p\mathcal{A}p \otimes K(H)) &= M((p \otimes 1)(\mathcal{A} \otimes K(H))(p \otimes 1)) \\ &= (p \otimes 1)M(\mathcal{A} \otimes K(H))(p \otimes 1) \end{aligned}$$

is a W^*-algebra. Therefore by Lemma 2.3, p is finite rank. □

The non unital version of the Mingo's lemma follows.

Corollary 2.7. *Suppose that H is a separable infinite dimensional Hilbert space and $\mathcal{A}$ is a C^*-algebra such that the multiplier algebra $M(\mathcal{A} \otimes K(H))$ is a W^*-algebra, then $\mathcal{A}$ is a C^*-algebra of compact operators.*

Proof. Since $\mathcal{A} \otimes K(H)$ is stable, it is a C^*-algebra of compact operators, and so is $\mathcal{A}$ by Lemma 2.5. □

It is well known that if $\mathcal{A}$ is a W^*-algebra and E is a selfdual Hilbert $\mathcal{A}$-module, then $B(E)$ is a W^*-algebra. The converse is not true, as for $E = \mathcal{A} = c_0$, $B(E) = \ell^\infty$ is a W^*-algebra [19]. However, if $\mathcal{A}$ is a C^*-algebra of compact operators on some Hilbert space, then $B(E)$ is a W^*-algebra, for every Hilbert $\mathcal{A}$-module E [6]. Here we show the converse.

Recall that the C^*-algebra $K(E)$ of compact operators on E is generated by rank one operators $\theta_{\xi,\eta}(\zeta) = \xi\langle\eta,\zeta\rangle$, for $\xi,\eta \in E$, and the multiplier algebra $M(K(E))$ is isomorphic to $B(E)$. Also, if H is a separable infinite dimensional Hilbert space, then $E = H \otimes \mathcal{A}$ is a Hilbert C^*-module over $\mathbb{C} \otimes \mathcal{A} = \mathcal{A}$, denoted by $H_{\mathcal{A}}$. It plays an important role in the theory of Hilbert C^*-modules.

Theorem 2.8. *For any C^*-algebra $\mathcal{A}$, the following are equivalent:*
(i) $\mathcal{A}$ is a C^*-algebra of compact operators,
(ii) $B(E)$ is a W^*-algebra, for each Hilbert $\mathcal{A}$-module E,
(iii) $B(H_{\mathcal{A}})$ is a W^*-algebra.

Proof. It is enough to show that (iii) implies (i). Since

$$K(H_{\mathcal{A}}) = K(H \otimes \mathcal{A}) \cong K(H) \otimes K(\mathcal{A}) = K(H) \otimes \mathcal{A}$$

we have $B(H_{\mathcal{A}}) \cong M(K(H) \otimes \mathcal{A})$. By assumption, $B(H_{\mathcal{A}})$ is a W^*-algebra and so $\mathcal{A}$ is a C^*-algebra of compact operators by Corollary 2.7.

□

J. Schweizer in [20] remarked that for a C^*-algebra $\mathcal{A}$, some problems on Hilbert $\mathcal{A}$-modules can be reformulated as problems on right ideals of $\mathcal{A}$, since submodules of a full Hilbert $\mathcal{A}$-module are in a bijective correspondence with the closed right ideals of $\mathcal{A}$. Therefore, one may wonder if the previous result could be reformulated in the language of right ideals. Actually, if I is a (closed) right ideal of $\mathcal{A}$, then I is a right Hilbert $\mathcal{A}$-module with inner product $\langle a, b\rangle = a^*b$, for $a, b \in I$, and in this case, $K(E)$ equals to the hereditary C^*-algebra $I \cap I^*$ and so $B(E) = M(I \cap I^*)$. Therefore, one may expect that C^*-algebras $\mathcal{A}$ of compact operators may be characterized by the property that for every hereditary C^*-subalgebra $\mathcal{B}$ of $\mathcal{A}$, $M(\mathcal{B})$ is a W^*-algebra.

Unfortunately, this is not the case for non separable C^*-algebras, as the following counterexample shows. However, if $\mathcal{A}$ is separable and p is a projection as in the proof of Theorem 2.6, then $p\mathcal{A}p$ is a separable W^*-algebra, hence finite dimensional (also see Theorem 2.8 in [2]).

Example 2.9. For the Stone-Cech compactification $\beta\mathbb{N}$ of the natural numbers, the algebra of continuous functions $C(\beta\mathbb{N})$ is a W^*-algebra. Let x be any point of $\beta\mathbb{N}$ that is not a natural number and let $\mathcal{A}$ be the C^*-subalgebra of $C(\beta\mathbb{N})$ consisting of those functions vanishing at x. Let $\mathcal{B}$ be a hereditary C*-subalgebra of $\mathcal{A}$ (which is an ideal, since $\mathcal{A}$ is abelian). Then there is an open subset U of $\beta\mathbb{N}$ such that $\mathcal{B}$ consists of functions in $\mathcal{A}$ that vanish outside U. Let V be the closure of U. Then V is also open. For every $c \in C(V)$ we may extend c by zero outside V, and thereby view $C(V)$ as a W^*-subalgebra of $C(\beta\mathbb{N})$. Observe that $M(\mathcal{B}) = C(V)$: clearly $\mathcal{B}$ is an ideal in $C(V)$, so it suffices to note that for any $0 \neq c \in C(V)$, $c\mathcal{B} \neq 0$. To see this, we note that c is non-zero on a nonvoid open subset W of V, hence $W \cap U \setminus x$ is a nonvoid open set. Hence there exists a non-zero continuous function b with support in $W \cap U \setminus x$. Thus $b \in B$ and $cb \neq 0$. Therefore $M(\mathcal{B}) = C(V)$ is a W^*-algebra, but $\mathcal{A}$ cannot be a C^*-algebra of compact operators.

3. Morita equivalence

The notion of (strong) Morita equivalence of C^*-algebras was introduced by M. Rieffel in [19]. Two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are *(strongly) Morita equivalent* if there is an $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{M}$, which is a left full

Hilbert C^*-module over $\mathcal{A}$, and a right full Hilbert C^*-module over $\mathcal{B}$, such that the inner products ${}_{\mathcal{A}}\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{\mathcal{B}}$ satisfy ${}_{\mathcal{A}}\langle x,y\rangle z = x\langle y,z\rangle_{\mathcal{B}}$ for all $x,y,z \in \mathcal{M}$. Such a module $\mathcal{M}$ is called an $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule.

It would be interesting to investigate those properties of C^*-algebras which are preserved under Morita equivalence. These include, among other things nuclearity, being type I, and simplicity [3, 7, 12, 17, 18, 21, 22]. Now if one of the two Morita equivalent C^*-algebras is a W^*-algebra, it is natural to ask if so is the other. The answer to this question, as it posed is obviously negative, as Hilbert space H is a $K(H)$-$\mathbb{C}$-imprimitivity bimodule, and so C^*-algebras $K(H)$ and $\mathbb{C}$ are Morita equivalent. However we may rephrase that question in the following less trivial form.

Question 3.1. Suppose that C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent and the C^*-algebra $M(\mathcal{A})$ is a W^*-algebra, is it then true that $M(\mathcal{B})$ is a W^*-algebra?

By Theorem 2.8, we can show that the above property holds for C^*-algebra $\mathcal{A}$ exactly when $\mathcal{A}$ is a C^*-algebra of compact operators. In fact, we have the following result.

Theorem 3.2. *Let $\mathcal{A}$ be a C^*-algebra such that $M(\mathcal{B})$ is a W^*-algebra, for any C^*-algebra $\mathcal{B}$ which is Morita equivalent to $\mathcal{A}$. Then $\mathcal{A}$ is a C^*-algebra of compact operators.*

Proof. Let $\mathcal{B} = K(H_{\mathcal{A}})$. Since $H_{\mathcal{A}}$ is a full Hilbert $\mathcal{A}$-module, then $\mathcal{B}$ is Morita equivalent to $\mathcal{A}$. By assumption, $B(H_{\mathcal{A}}) \cong M(\mathcal{B})$ is a W^*-algebra, hence $\mathcal{A}$ is a C^*-algebra of compact operators, by Theorem 2.8. □

However, we give an affirmative answer to the above question, when both C^*-algebras are unital.

Recall that a Hilbert C^*-module E on a C^*-algebra $\mathcal{A}$ is called *self dual* if for every bounded linear $\mathcal{A}$-module map $\varphi : E \to \mathcal{A}$ there is an element $y \in E$ such that $\varphi(\cdot) = \langle y,\cdot\rangle$.

Lemma 3.3. *Let E be a right Hilbert C^*-module over a C^*-algebra $\mathcal{A}$ such that $K(E)$ is unital. then*
(i) E is self dual.
(ii) $B(E)$ is a W^*-algebra, whenever $\mathcal{A}$ is a W^*-algebra.

Proof. By hypothesis there are elements $x_1,\cdots,x_n$ and $y_1,\cdots,y_n$ in E such that $\sum_{i=1}^{n}\theta_{x_i,y_i} = 1 \in K(E)$. Thus, for every bounded linear

$\mathcal{A}$-module map $\varphi : E \to \mathcal{A}$ and $x \in E$ we have

$$\varphi(x) = \varphi(\sum_{i=1}^{n} \theta_{x_i,y_i} x) = \varphi(\sum_{i=1}^{n} x_i \langle y_i, x\rangle) = \sum_{i=1}^{n} \varphi(x_i)\langle y_i, x\rangle$$
$$= \sum_{i=1}^{n} \langle y_i \varphi(x_i)^*, x\rangle = \langle \sum_{i=1}^{n} y_i \varphi(x_i)^*, x\rangle.$$

Therefore $\varphi(x) = \langle y, x\rangle$, where $y = \sum_{i=1}^{n} y_i \varphi(x_i)^*$. Hence E is selfdual.
Now (ii) follows from (i) and [16, Proposition 3.10]. □

Now if E is an $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule, then $\mathcal{A} \cong K_{\mathcal{B}}(E)$ and $\mathcal{B} \cong K_{\mathcal{A}}(E)$. Therefore, the following partial answer to the above question follows from the above lemma.

Theorem 3.4. *Suppose that unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent. Then $\mathcal{A}$ is a W^*-algebra if and only if $\mathcal{B}$ is a W^*-algebra.*

A similar result can be proved for operator algebras. Let $\mathcal{A}$ and $\mathcal{B}$ be operator algebras. We say that $\mathcal{A}$ and $\mathcal{B}$ are (strongly) Morita equivalent if they are Morita equivalent in the sense of Blecher, Muhly, Paulsen [8]. In [8], it is proved that two C^*-algebras are (strongly) Morita equivalent (as operator algebras) if and only if they are Morita equivalent in the sense of Rieffel.

Theorem 3.5. *Suppose that unital operator algebras $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent. Then $\mathcal{A}$ is a dual operator algebra if and only if $\mathcal{B}$ is a dual operator algebra.*

Proof. Let $\pi : \mathcal{A} \to B(H)$ be a completely isometric normal representation of $\mathcal{A}$ on some Hilbert space H. Then there exist a completely isometric representation $\rho : \mathcal{B} \to B(K)$ of $\mathcal{B}$ on a Hilbert spaces K and subspaces $X \subseteq B(K, H)$, $Y \subseteq B(H, K)$ such that

$$\pi(\mathcal{A})X\rho(\mathcal{B}) \subseteq X,\ \rho(\mathcal{B})Y\pi(\mathcal{A}) \subseteq Y,\ \pi(\mathcal{A}) = \overline{XY}^{\|\cdot\|},\ \rho(\mathcal{B}) = \overline{YX}^{\|\cdot\|}$$

Since π is normal, we have $\pi(\mathcal{A}) = \overline{\pi(\mathcal{A})}^{w*}$. Now $X\rho(\mathcal{B})Y \subseteq \pi(\mathcal{A})$ implies that $X\overline{\rho(\mathcal{B})}^{w*}Y \subseteq \pi(\mathcal{A})$. Therefore

$$YX\overline{\rho(\mathcal{B})}^{w*}YX \subseteq Y\pi(\mathcal{A})X \subseteq \rho(\mathcal{B}),$$

and so $\rho(\mathcal{B})\overline{\rho(\mathcal{B})}^{w*}\rho(\mathcal{B}) \subseteq \rho(\mathcal{B})$. Since $\rho(\mathcal{B})$ is a unital algebra we have $\overline{\rho(\mathcal{B})}^{w*} \subseteq \rho(\mathcal{B})$, hence $\overline{\rho(\mathcal{B})}^{w*} = \rho(\mathcal{B})$. Therefore $\mathcal{B}$ is a dual operator algebra.

□

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